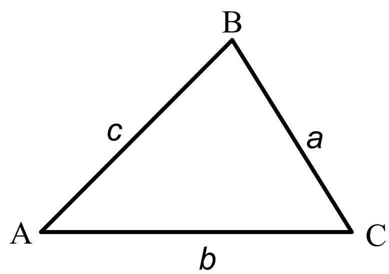
Proof of the Law of Cosines

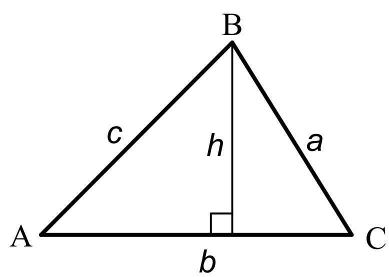
Take a triangle. Any triangle. Give the sides and angles **names**, so we can talk about them. Angles are named with an uppercase letter, because the vertex is a *point*; sides are named with lowercase letters, because they are (parts of) *lines*.

Traditionally, math people use the SAME LETTER to name an angle and the side **opposite** that angle – but the angle is the uppercase letter, and the (opposite) side is the lowercase letter. Like this:

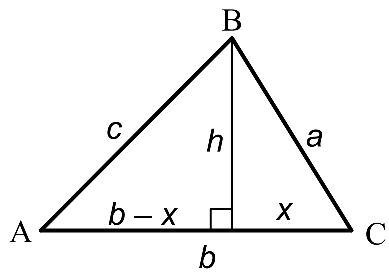


Notice that side ***a*** is opposite angle ***A***; side ***b*** is opposite angle ***B***; etc.

Now we have a triangle; we can draw in an ***altitude*** from one vertex to the opposite side, **perpendicular** to that side. I'll label that altitude ***h***, for height. This creates two right triangles inside the original triangle. See below:



That also creates a new point (which we won't name, because we don't need to talk about it). That point also divides that side into two pieces, which we'll need to talk about, so we have to name them. We'll call one piece ***x***, and then the other piece can be ***b* – *x***, because the original side was named ***b*** already. Below:



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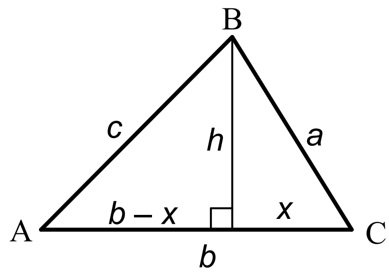
Now we're ready to do some Pythagorean Theorem and some algebra.

Notice the two right triangles. We can use the Pythagorean Theorem twice to figure out relationships among the lengths of those segments:



Let's use the Pythagorean Theorem. Twice.

$c^2 = (b-x)^2 + h^2$ and $a^2 = x^2 + h^2$

Rearrange both equations to isolate h^2:

$c^2 - (b-x)^2 = h^2$ and $a^2 - x^2 = h^2$

Both $c^2 - (b-x)^2$ and $a^2 - x^2$ equal h^2, so they must equal each other:

$c^2 - (b-x)^2 = a^2 - x^2$

expand $(b-x)^2$

$c^2 - (b^2 - 2bx + x^2) = a^2 - x^2$

distribute that negative sign in front of $(b^2 - 2bx + x^2)$

$c^2 - b^2 + 2bx - x^2 = a^2 - x^2$

Look! We got $-x^2$ on both sides! So add x^2 to both sides:

$c^2 - b^2 + 2bx = a^2$

Now let's get c^2 all by itself:

$c^2 = a^2 + b^2 - 2bx$

What is x? Look at the picture; $\frac{x}{a} = \cos C$, so $x = a\cos C$

So we can substitute $a\cos C$ for x

$c^2 = a^2 + b^2 - 2ba\cos C$

Put a and b in alphabetical order, and ...

$c^2 = a^2 + b^2 - 2ab\cos C$

Hey! It's the Law of Cosines!